

On Mathematical Imagination

From whole numbers to infinity—and beyond

by ROBERT KAPLAN and ELLEN KAPLAN

BOOKS

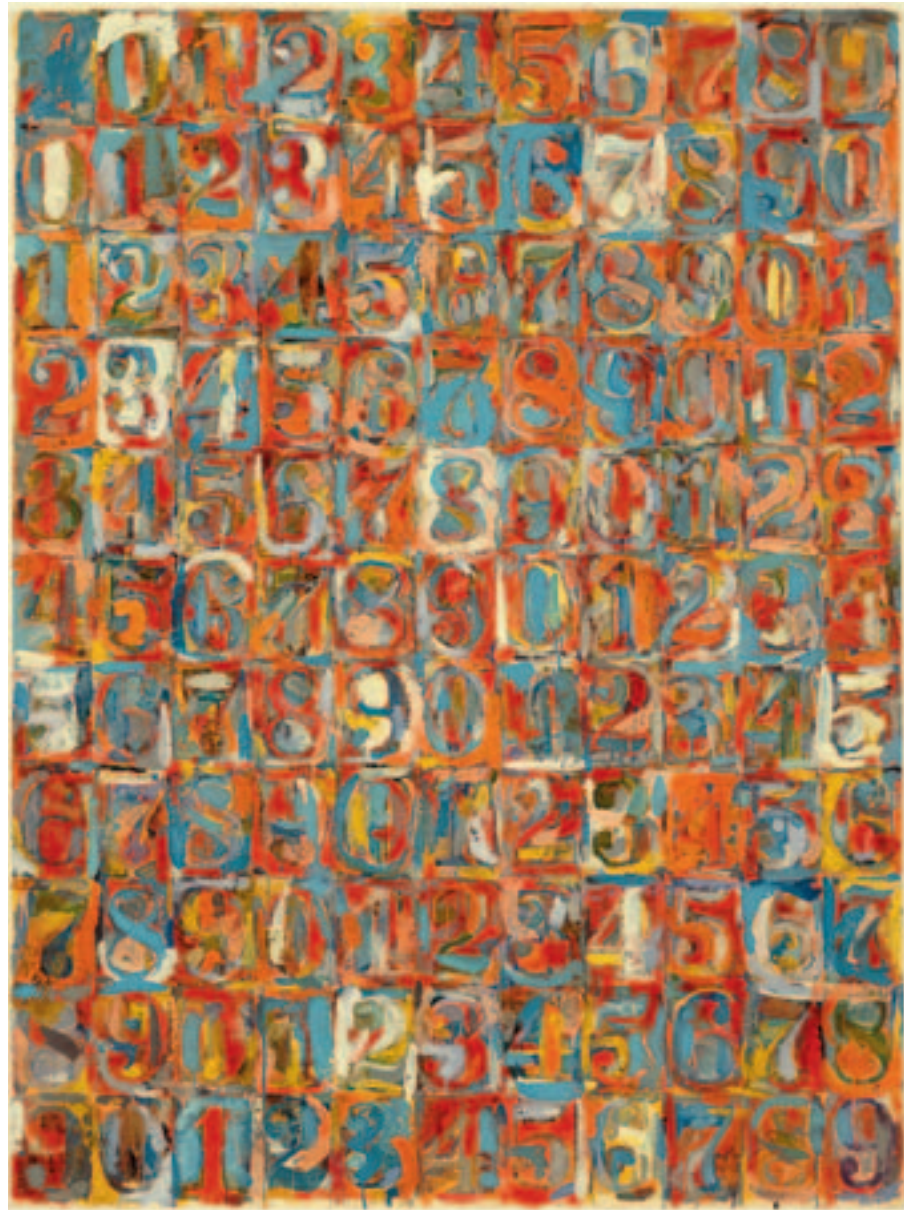
THE LUNATIC, the lover, and the poet, said Shakespeare, are of imagination all compact. He forgot the mathematician, whose daily concerns are

shapes in 27 dimensions, series that converge after more terms than there are particles in the universe, numbers larger than infinity, and others infinitesimally small as well as surreal and hyperreal and imaginary. The most monstrous thing about these fantasies, of course, is that they turn out to describe how our one and only universe works.

The people who intermediate between lunatics and the world used to be called *alienists*; the go-betweens for mathematicians are called *teachers*. Many a student may rightly have wondered if the terms shouldn't be reversed. Those wonderful outworks of the imagination that mathematicians produce rarely make it to the classroom, where mind-numbing drill in what seems a dead language tends to rule instead.

Benedict Gross '71, Ph.D. '78, and Joseph D. Harris '72, Ph.D. '78, *The Magic of Numbers* (Prentice Hall, \$53.33). Barry Mazur, Jf '62, *Imagining Numbers* (particularly the square root of minus fifteen) (Farrar, Straus and Giroux, \$22).

We are, however, at a fulcrum moment in the history of math teaching, with the balance tipping at last from oases of horror in a desert of boredom to epiphanies that make mathematics as appealing as is its sister-art, music. Outstanding examples of this revolution are *The Magic*



of Numbers, by Benedict H. Gross and Joseph D. Harris (based on their Core course in quantitative reasoning) and *Imagining Numbers* (particularly the square root of minus fifteen), by Gade University Profes-

sor Barry Mazur—two books intended to draw lay readers into the secrets of what mathematics is really like and how our fellow humans invent it.

Gross, who is Leverett professor of

A Succinct Credo

Squash aficionados—and Harvard has nurtured many of them, at first under legendary coach Harry Cowles—will find deeply satisfying and often Crimson-colored the book *Squash: A History of the Game*, by James Zug, with a foreword by the late George Plimpton '48 (Scribner, \$30). Zug describes Cowles as “the Knute Rockne of squash...the first squash genius. He created the prototype of future squash coaches: the excellent player, the genial taskmaster, the quiet technician, the unflappable leader. He also changed the nature of squash in America.... In 14 seasons, Cowles laid his hands on a stream of teenagers and trophies appeared. He coached seven national champions, more than any other person in history.” In 1936 he had a breakdown; he was committed to a mental hospital and diagnosed with schizophrenia. Later he had a frontal lobotomy and suffered mightily until his death in 1958. But this, writes Zug, is how he was treasured in memory by those he coached:

THEY REMEMBERED HIM with a tournament in New York that bore his name; after his death it was not renamed the Harry Cowles Memorial, because his players always felt he was with them, alive or dead. They remembered his love for squash. They remembered his loyalty to Harvard, how he turned down offers to coach at other schools for twice the salary. They remembered his brilliant drop shots. They remembered his quiet counsel. They remembered one winter afternoon driving on a windswept back road in New England, five players crammed into Cowles's car, coming home from a match. They were discussing, as undergraduates are apt to do on long Cimmerian drives in the bleak midwinter, theoretical matters relating to squash. The issue this time was mental toughness. It was the Great Depression. Hope was fighting a losing battle to fear. How much courage did a person need? What breaks down determination and endurance? When should one quit when an opponent is clearly better? After a while, the players paused. [Germain] Glidden turned to Cowles and asked how long a player should battle even if it is hopeless. Everyone went silent. They could hear the murmur of the engine, the clank of



Harry Cowles

THE SECOND H BOOK OF HARVARD ATHLETICS

the car heater, the susurrous crush of snow under the wheels. Cowles stared straight ahead at the darkening road and said, “You should fight until you're dead.”

mathematics and dean of Harvard College, and Harris, who is Higgins professor of mathematics, have managed a miracle: addressing an audience they rightly assume to be resistant, they lead it gently from utter scratch (how many whole numbers are there from 1 to 10?) to counting with Catalan numbers (in how many ways can n pairs of parentheses appear in a sentence?)—and they do this with bonhomie, grace, and humor. Can you imagine any textbook speaking to you like this?

Let's suppose you climb out of bed one morning, still somewhat groggy from the night before. You grope your way to your closet, where you discover that your cache of clean clothes has been reduced to four shirts and three pairs of pants. It's far too early to exercise any aesthetic judgment whatsoever: any shirt will go with any pants; you only need something that will get you as far as the dining hall and that blessed, life-giving cup of coffee. The question is,

How many different outfits can you make out of your four shirts and three pairs of pants?

Admittedly the narrative took a sharp turn toward the bizarre with that last sentence. Why on earth would you or anyone care how many outfits you can make? Well, bear with us while we try to answer it anyway.

This is a far cry from the standard word-problems that the Canadian humorist Stephen Leacock made such wonderful fun of: dashing old A rowing the best boat while B, as always, has all he can do to keep his leaky tub afloat—and you ask who won?

The Magic of Numbers takes us from counting to the heart of mathematics—number theory—where we feel the pulse and see the valves opening and closing. Like medical students, the lucky people in the Gross and Harris class, and now all readers, are shown the deepening layers of anatomy, the sense the systems make, and how they function; then, in carefully calibrated exercises, they have a chance to do probing of their own. What was impersonal becomes part of each reader's personality. The motto here could well be what medical students are told about operations: “Watch one, do one, teach one.”

The book leads you so far toward the frontiers of mathematics that no one could be blamed for losing sight of everything else and waking up 20 years later in

its mountains. But just when our heads begin to break through the clouds, Gross and Harris return us to a heightened reality by showing how relevant what we have learned is to a vital practical issue: public-key cryptography. Here the

primes, whose tantalizing mysteries Gross and Harris have explored with their students, act as the covert keys to encrypted messages that are openly transmitted along with a public key: the product of two very large primes known

only to the receiver. Why, then, can't a spy decipher the message? Because to work back from the product to its prime factors would take more time (even with the fastest computers in the world) than the enemy has to spare—and so a problem in the surreal world of espionage is solved by recourse to the more than real world of mathematics. We find ourselves initiated now in what, at the book's beginning, we knew about only by rumor.

Gross and Harris close their book by saying, "If you've stayed with us till now, you know a tremendous amount of real mathematics—more than all but a small fraction of people walking around.... [N]ow it's time for...you to close the book and get on with your life." What they don't say is that your life will be wholly different. Pindar praised the winners of Olympic events by telling them they had managed, once only and briefly, to touch the bronze sky. Those who win through to the end of *The Magic of Numbers* will be for the rest of their lives in touch with the accessible mystery of things.

AND WHICH of those mysteries is more perplexing than the numbers once dismissed even by mathematicians themselves as "fictitious," "sophistical," and—the name which has stuck—"imaginary":

Perhaps most perplexing are the numbers mathematicians called "fictitious," "sophistical," and "imaginary."

those numbers whose squares are negative (so that if i stands for $\sqrt{-1}$, $i^2 = -1$). If you grant them citizenship in the Republic of Numbers, you won't be able to complain if your neighbor's field, which is $3i$ yards wide by $7i$ yards long, has an area of -21 square yards (but then, why should you complain, since that area takes away nothing from yours, and you won't even see his rows of fanciful potatoes?). Barry Mazur's superb excursion into *Imagining Numbers* shows us much more than the troubled history of this idea and the cluster of daring, ingenious, and at times out-

rageous people who drove it along. He takes us to the central activity of mathematics—which is imagining—and asks himself and us: "Exactly what is it?"

Mazur doesn't expect his readers to have done any mathematics recently or even to remember what they were exposed to in school. "This book," he explains, "is written for people who may wish to experience an act of mathematical imagining and to consider how such an experience compares with the imaginative work involved in reading and understanding a phrase in a poem."

What follows is a fugue whose subject is mathematics and whose countersubject is poetry. Each supports, alters, and harmonizes with the other. We find ourselves wondering about very basic mental actions: what do we *experience* when we read "the yellow of the tulip"? What allows—even encourages—us to think that adding two numbers and then multiplying the sum by a third number should give the same result as multiplying each by that third and then adding the outcomes? We

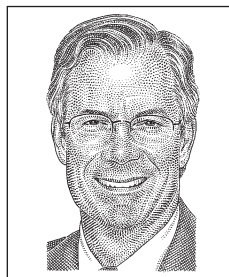
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are here on the mind's floor, fitting together the shadows cast by the world.

A fugue can't be summarized: it must be listened to and played. But we can promise readers of Mazur's work that they will find in it the same sort of transformations that give music its vitality and depth. Numbers change from objects to actions. Our thinking about them changes from algebraic to geometric. Imagination is no longer visualizing but intuiting. Answers become questions. In his contemplation of modern poetry and his coming to grips with how Renaissance mathematicians

made sense of their strange new numbers, it is always the internal structure of the act of imagining that Mazur arrows toward.

THIS SHIFT OF EMPHASIS from memorizing to imagining is key to the change we spoke of in how mathematics—this art of the infinite—is taught. In *The Math Circle*, which we began at Harvard in 1994 to teach people of all ages about the enjoyment of mathematics, we never tell our students anything; instead we let them explore the near and far reaches in congenial conversation. As in *The Magic of Numbers*, our

style is collegial: whether the students are six or 60, we all talk as fellow adventurers and we let any conjecture run until it self-corrects or blossoms into a proof. As in *Imagining Numbers*, specific problems always play out on general backgrounds: why does it matter—to math, to mind, to understanding the world—that the medians of a triangle are concurrent, or that there are consecutive primes with a gap between them as large as you'd like; and how does the hierarchy of ever larger infinities bear on our finite lives?

Our students can't get enough of math when it is learned without pressure of time or performance. We started with 29, nine years ago, and have more than 200 now; this past year we began teaching our approach to teachers in a Harvard Extension School course. We take whoever comes, the math phobic and math friendly alike, and by working together

CHAPTER & VERSE

A correspondence corner for not-so-famous lost words

John Glismann would like to learn the source of "Here's to the man who'll take my place when it's time for me to go."

John Havelock writes, "In early 1968, while working for his administration, I got my first tip-off that LBJ was not going to run again when he made a lightly reported speech, perhaps in a church, [with] lines to the effect that his heart was heavy and 'the springtime of man seems farther and farther away.' I have wondered [about] the full quotation and whether it was Lyndon's invention or came from an earlier source." Can a reader answer his question?

Cyrus Mody requests "the origin, or interesting uses, of 'moot for the nonce.'"

Addison Merrick seeks a source for "Memory is the only paradise man cannot be expelled from."

"upside-down format" (November-December 2003). Linda Liu Behar was first to suggest *Happenstance* (1993), by Carol Shields; William Batchelder first identified *The Loving Couple* (1956), by Virginia Rowans [Patrick Dennis]; Carol Buckley suggested *Say It with Bricks: A Few Remarks about Husbands/Say It with Oil: A Few Remarks about Wives* (1923), by Nina Wilcox Putnam and Ring Lardner; and Erik Ghenoïu suggested *The Inner Side of the Wind, or the Novel of Hero and Leander*, by the Serbian author Milorad Pavic (1991; 1993 in English translation).

"amnesiac ex-girlfriend" (November-December 2003). Gail Banks was first to recognize the reference to Margot Livesey's novel *The Missing World* (2000).

"Marriage...a bag of snakes" (November-December 2003). Charles Maurer traced the remark to Sir Thomas More's *Dialogue concerning heresies and matters of religion* (1528; bk 1, ch. 26): "I have heard my father [Sir John More] merrily say every man is at the choice of his wife, that you should put your hand into a blind bag full of snakes and eels together, seven snakes for one eel, you would I ween reckon it a perilous choice to take up one at adventure though you had made your special prayer to speed well." The text appears in *Saint Thomas More* (1957), by E.E. Reynolds. In John Campbell's *Lives of the Lord Chancellors* (1846), Maurer added, a paraphrase runs: "He would compare the multitude of women which are to be chosen for wives unto a bag full of snakes, having among them a single eel: now, if a man should put his hand into this bag, he may chance to light on the eel; but it is a hundred to one he shall be stung by a snake."

"Oblivion" (November-December 2003). Anthony Shipps was the first to identify the last stanza of Ralph Waldo Emerson's poem "Waldeinsamkeit."

Send inquiries and answers to "Chapter and Verse," Harvard Magazine, 7 Ware Street, Cambridge 02138.

Our students can't get enough of math when it is learned without pressure of time or performance.

their common humanity broadens. Why won't the young leave at the end of a late afternoon class? Why will adolescents get up early on Sunday mornings? Because by discovering mathematics for oneself—as these two books encourage their readers to do—one owns the results. Property, as Locke pointed out, is that with which you have commingled your labor.

The Math Circle, *The Magic of Numbers*, and *Imagining Numbers* are part of a New Instauration that will bring mathematics, at last, into its rightful place in our lives: a source of elation through the questions it puts to us and the answers we shape. ▽

Robert Kaplan, A.M. '69, and Ellen Kaplan '57, A.M. '59, of Cambridge, are the founders of The Math Circle, which is now poised, they report, to expand nationally and internationally. They are the authors of *The Art of the Infinite: The Pleasures of Mathematics*. Robert Kaplan has also written *The Nothing That Is: A Natural History of Zero*.